Modal Information Logic of Incomparable Fusions

Søren Brinck Knudstorp
Extract from MSc thesis, supervised by Johan van Benthem and Nick Bezhanishvili

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Universiteit van Amsterdam

Outline of the talk

- Introduction and motivation
- Proof (outline) of main theorem
- Conclusion

Defining the basic modal information logics (MILs)

Definition (language and semantics)

The language is given by

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle \min \rangle \varphi\psi,$$

and the semantics of '$\langle \min \rangle$' is:

$$w \Vdash \langle \min \rangle \varphi\psi \quad \text{iff} \quad \exists u, v (u \Vdash \varphi;\ v \Vdash \psi; \ w \in \min\{u, v\})$$

Example

$w, w' \Vdash \langle \min \rangle pq$, but
$w, w' \nVdash \langle \sup \rangle pq$

Definition (frames and logics)

- A poset frame is a pair $(W, \leq)$, where W is a set and $\leq$ is a partial order on W (i.e., refl., tran., and anti-symm.).
- Depending on the interpretation of the modality, we get two logics:
  - MIL^{Min}, our modal information logic of incomparable fusions;
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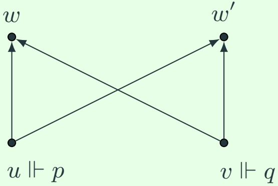

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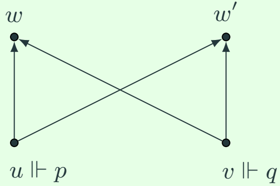

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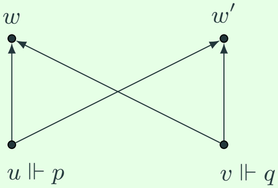

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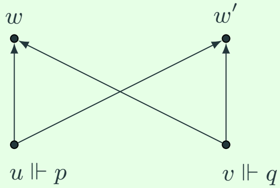

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Why MILs?

1. Introduced to model a theory of information (by van Benthem (1996))
2. Modestly extend $\mathbf{S4}$

Why *minimal* upper bounds?

1.' Formalizes (informational) settings in which states can have multiple incomparable 'fusions'

2.' The resulting logic modestly extends $\mathbf{S4}$

3. But primarily, motivated by technical/mathematical curiosity:

Knudstorp (Forthcoming) axiomatizes *MIL, and its completeness proof relies heavily on this distinction between* ***minimal*** *and* ***least*** *upper bounds.*

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(R) Figuring out how MIL^{Min} and MIL relate;

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It seems that one should, at least, expect

$$MIL \neq MIL^{Min}$$

However, the main concern for the rest of the talk is to show that, in fact, $MIL = MIL^{Min}$

Proof of $\mathit{MIL} \subseteq \mathit{MIL}^{\mathit{Min}}$

Our starting point is the following result:

Axiomatization of *MIL* [Knudstorp (Forthcoming)]

MIL is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \to \langle \text{sup} \rangle pq$

(4) $PPp \to Pp$

(Co.) $\langle \text{sup} \rangle pq \to \langle \text{sup} \rangle qp$

(Dk.) $(p \wedge \langle \text{sup} \rangle qr) \to \langle \text{sup} \rangle pq$

Using this we get:

Proposition

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Proof.

Routine check that $\mathit{MIL}^{\mathit{Min}}$ is a normal modal logic validating (Re.), (4), (Co.), (Dk.). □

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Proof of $MIL \supseteq MIL^{Min}$: corollaries and framework

It remains to show that

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Note that this would also allow us to deduce:

Corollary (Axiomatization and Decidability)

MIL^{Min} is decidable and axiomatized as shown before (because MIL is [cf. Knudstorp (Forthcoming)]).

Framework for proof of $MIL \supseteq MIL^{Min}$.

- Suppose that $\varphi \notin MIL$.
- Then $\mathbb{M}^S, w \nVdash \varphi$ for some supremum-model $\mathbb{M}^S$.
- **Idea:** Transform $\mathbb{M}^S$ into a minimum-model $\mathbb{M}^M$ s.t. $\mathbb{M}^M, w \nVdash \varphi$.

Formally, the proof goes by representation using onto p-morphisms.

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Observation: Given a partial order '$\leq$': $w' \in \min\{u', v'\} \Leftarrow w' = \sup\{u', v'\}$

Recall: We want to mend $\mathbb{M}^S \rightsquigarrow \mathbb{M}^M$ (in a satisfaction-preserving way).

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Observation: There are two ways for an upper-bounded set $\{u, v\}$ to not have a supremum:

(i) Incomparable upper bounds; and

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Main idea: Transform all instances of (i) in $\mathbb{M}^S$ into instances of (ii)!

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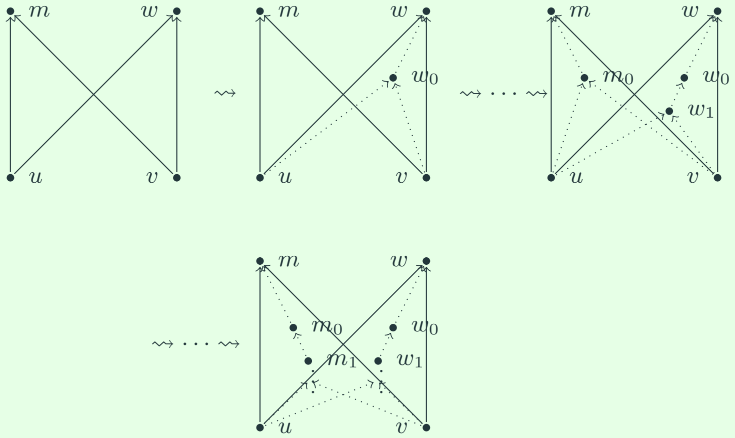
Proof of $MIL \supseteq MIL^{Min}$: naive transformation

From (i) to (ii): naive idea



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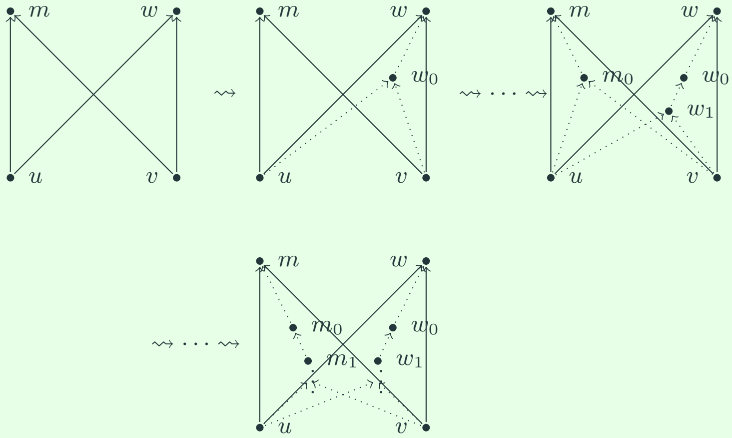

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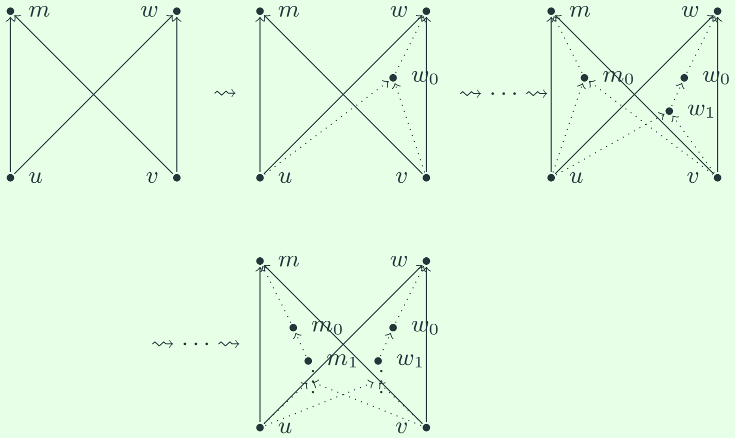

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Solution 2: For x to stay supremum of $\{y, z\}$, we must make x see w_0 (and w_1, etc.). In general, the least downset containing $\{u, v\}$ and closed under binary suprema should see w_0 (and w_1, etc.).

Using this transformation essentially allows us to prove the following:

Principal lemma

Let $(W, \leq)$ be a poset frame and $\{w, u, v\} \subseteq W$ s.t. $w \in \min\{u, v\}$ but $w \neq \sup\{u, v\}$. Then $(W, \leq)$ is the p-morphic image (w.r.t. the supremum relation) of a poset frame $(W', \leq')$ s.t.

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Proof of $MIL \supseteq MIL^{Min}$: final steps

Proposition (representation)

Every poset frame $(W, \leq)$ is the p-morphic image (w.r.t. its supremum relation) of a poset frame $(W', \leq')$ satisfying

$$\forall w', v', u' \in W' \left(w' \in \min\{u', v'\} \Leftrightarrow w' = \sup\{u', v'\}\right).$$

Proof idea.

Use the preceding lemma to iteratively resolve all failures of

$$w' \in \min\{u', v'\} \Rightarrow w' = \sup\{u', v'\}. \quad \square$$

Thus, we have concluded our proof of $MIL = MIL^{Min}$; in this setting, the two interpretations cannot be told apart.

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This raises the question: when can we tell the interpretations apart?

Telling apart the $\langle \min \rangle$ and $\langle \sup \rangle$ interpretations

From partial orders to preorders:

- Our logics are defined over *posets*:

$$MIL_{Pos} := MIL, \qquad MIL^{Min}_{Pos} := MIL^{Min}$$

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- What happens if we generalize to preorders $MIL_{Pre}, MIL^{Min}_{Pre}$?

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- What happens if we add the implication '\' with semantics

$$v \Vdash \varphi \backslash \psi \quad \text{iff} \quad \forall u, w([u \Vdash \varphi, w = \sup\{u, v\} / w \in \min\{u, v\}] \Rightarrow w \Vdash \psi)$$

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- Answer: They come apart! B/c $MIL \not\ni (Pp \wedge Pq) \to P\langle \sup \rangle pq \in MIL^{Min}$

Summary and main themes:

- Proved that $\langle \sup \rangle$ and $\langle \min \rangle$ interpretations result in the same logic $MIL = MIL^{Min}$.
  - Showed that MIL^{Min} is sound w.r.t. to axiomatization of MIL.
  - Collapsed the minimum-relation into the supremum-relation s.t.

$$w' \in \min\{u', v'\} \Leftrightarrow w' = \sup\{u', v'\}$$

    by transforming instances of (i) into instances of (ii).
- Obtained axiomatization and decidability for free.
- Considered other settings uncapable of distinguishing the interpretations: preorders and augmenting with '\'; but noted that the induced logics do come apart on finite structures.

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Thank you!

References I

Knudstorp, S. B. (Forthcoming). **“Modal Information Logics: Axiomatizations and Decidability”.** In: *Journal of Philosophical Logic* (cit. on pp. 8–16, 19–28).

Van Benthem, J. (1996). **“Modal Logic as a Theory of Information”.** In: *Logic and Reality. Essays on the Legacy of Arthur Prior.* Ed. by J. Copeland. Clarendon Press, Oxford, pp. 135–168 (cit. on pp. 8–16).

The principal lemma

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Proof.

Let $W' := W \sqcup {\downarrow}w = \{(x, 0), (y, 1) \mid x \in W, y \in {\downarrow}w\}$, and

$$f : W' \to W, (x, i) \mapsto x$$

For all $(x, i), (y, j) \in W'$, we let $(y, j) \leq' (x, i)$ **iff**

- $i = 0$ and $y \leq x$, **or**
- $j = i = 1$ and $y \leq x$, **or**
- $j = 0, i = 1, y \in A$ and $x = w$.

To show: (1) $(W', \leq')$ is a poset frame; (2) 1.-4. are satisfied; and (3) f is an onto p-morphism. □

Completeness of *MIL*: the basic idea

Example